

# On the size of graphs without repeated cycle lengths<sup>★</sup>

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## Abstract

In 1975, P. Erdős proposed the problem of determining the maximum number  $f(n)$  of edges in a graph with  $n$  vertices in which any two cycles are of different lengths. In this paper, it is proved that

$$f(n) \geq n + \frac{107}{3}t + \frac{7}{3}$$

for  $t = 1260r + 169$  ( $r \geq 1$ ) and  $n \geq \frac{2119}{4}t^2 + 87978t + \frac{15957}{4}$ . Consequently,  $\liminf_{n \rightarrow \infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2 + \frac{7654}{19071}}$ , which is better than the previous bounds  $\sqrt{2}$  [Y. Shi, Discrete Math. 71(1988), 57-71],  $\sqrt{2.4}$  [C. Lai, Australas. J. Combin. 27(2003), 101-105]. The conjecture  $\lim_{n \rightarrow \infty} \frac{f(n)-n}{\sqrt{n}} = \sqrt{2.4}$  is not true.

*Key words:* Graph, cycle, number of edges.

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## 1 Introduction

Let  $f(n)$  be the maximum number of edges in a graph on  $n$  vertices in which no two cycles have the same length. In 1975, Erdős raised the problem of

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determining  $f(n)$  (see Bondy and Murty [1], p.247, Problem 11). Shi [15] proved a lower bound.

**Theorem 1 (Shi [15])**

$$f(n) \geq n + [(\sqrt{8n - 23} + 1)/2]$$

for  $n \geq 3$ .

Chen, Lehel, Jacobson, and Shreve [3], Jia [5], Lai [6–8], Shi [16,18–20] obtained some additional related results.

Boros, Caro, Füredi and Yuster [2] proved an upper bound.

**Theorem 2 (Boros, Caro, Füredi and Yuster [2])** For  $n$  sufficiently large,

$$f(n) < n + 1.98\sqrt{n}.$$

Lai [9] improved the lower bound.

**Theorem 3 (Lai [9])**

$$f(n) \geq n + \sqrt{2.4}\sqrt{n}(1 - o(1))$$

Lai [6,9] proposed the following conjectures:

**Conjecture 4 (Lai [9])**

$$\lim_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} = \sqrt{2.4}.$$

**Conjecture 5 (Lai [6] )**

$$\liminf_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} \leq \sqrt{3}.$$

Markström [13] raised the following problem:

**Problem 6 (Markström [13])** Determine the maximum number of edges in a hamiltonian graph on  $n$  vertices with no repeated cycle lengths.

Results for the maximum number of edges in a 2-connected graph on  $n$  vertices in which no two cycles have the same length can be found in [2,3,15].

Survey articles on this problem can be found in Tian [21], Zhang [24], Lai and Liu [10].

The progress of all 50 problems in [1] can be find in Locke [12].

A related topic concerns the Entringer problem, which is to determine which simple graphs have exactly one cycle of each length  $l$ ,  $3 \leq l \leq v$  (see problem 10 in [1]). This problem was posed in 1973 by R. C. Entringer. For developments on this topic, see [4,11,13,14,17,22,23].

In this paper, we construct a graph  $G$  having no two cycles with the same length which leads to the following result.

**Theorem 7** Let  $t = 1260r + 169$  ( $r \geq 1$ ), then

$$f(n) \geq n + \frac{107}{3}t + \frac{7}{3}$$

for  $n \geq \frac{2119}{4}t^2 + 87978t + \frac{15957}{4}$ .

Hence conjecture 4 is not true.

## 2 Proof of theorem 7

**Proof.** Let  $t = 1260r + 169, r \geq 1$ ,  $n_t = \frac{2119}{4}t^2 + 87978t + \frac{15957}{4}$ ,  $n \geq n_t$ . We shall show that there exists a graph  $G$  on  $n$  vertices with  $n + \frac{107}{3}t + \frac{7}{3}$  edges such that all cycles in  $G$  have distinct lengths.

Now we construct the graph  $G$  which consists of a number of subgraphs:  $B_i$ ,  $(0 \leq i \leq 20t, 27t \leq i \leq 28t + 64, 29t - 734 \leq i \leq 29t + 267, 30t - 531 \leq i \leq 30t + 57, 31t - 741 \leq i \leq 31t + 58, 32t - 740 \leq i \leq 32t + 57, 33t - 741 \leq i \leq 33t + 57, 34t - 741 \leq i \leq 34t + 52, 35t - 746 \leq i \leq 35t + 60, 36t - 738 \leq i \leq 36t + 60, 37t - 738 \leq i \leq 37t + 799, i = 20t + j(1 \leq j \leq \frac{t-7}{6}), i = 20t + \frac{t-1}{6} + j(1 \leq j \leq \frac{t-7}{6}), i = 21t + 2j + 1(0 \leq j \leq t-1), i = 21t + 2j(0 \leq j \leq \frac{t-1}{2}), i = 23t + 2j + 1(0 \leq j \leq \frac{t-1}{2}), \text{ and } i = 20t + \frac{t-1}{6}, i = 20t + \frac{t-1}{3} + \frac{t-1}{6} - 1, i = 20t + \frac{t-1}{3} + \frac{t-1}{6}, i = 20t + \frac{2t-2}{3}, i = 21t - 2, i = 21t - 1, i = 26t).$

Now we define these  $B_i$ s. These subgraphs all only have a common vertex  $x$ , otherwise their vertex sets are pairwise disjoint.

For  $1 \leq i \leq \frac{t-7}{6}$ , let the subgraph  $B_{20t+i}$  consists of a cycle

$$xa_i^1 a_i^2 \dots a_i^{\frac{62t-8}{3}+2i} x$$

and a path:

$$xa_{i,1}^1 a_{i,1}^2 \dots a_{i,1}^{\frac{59t-5}{6}} a_i^{\frac{61t-1}{6}+i}$$

Based the construction,  $B_{20t+i}$  contains exactly three cycles of lengths:

$$20t + i, 20t + \frac{t-1}{3} + i - 1, 20t + \frac{2t-2}{3} + 2i - 1.$$

For  $1 \leq i \leq \frac{t-7}{6}$ , let the subgraph  $B_{20t+\frac{t-1}{6}+i}$  consists of a cycle

$$xb_i^1 b_i^2 \dots b_i^{\frac{62t-5}{3}+2i} x$$

and a path:

$$xb_{i,1}^1 b_{i,1}^2 \dots b_{i,1}^{10t-1} b_i^{\frac{61t-1}{6}+i}$$

Based the construction,  $B_{20t+\frac{t-1}{6}+i}$  contains exactly three cycles of lengths:

$$20t + \frac{t-1}{6} + i, 20t + \frac{t-1}{3} + \frac{t-1}{6} + i, 20t + \frac{2t-2}{3} + 2i.$$

For  $0 \leq i \leq t-1$ , let the subgraph  $B_{21t+2i+1}$  consists of a cycle

$$xu_i^1 u_i^2 \dots u_i^{25t+2i-1} x$$

and a path:

$$xu_{i,1}^1 u_{i,1}^2 \dots u_{i,1}^{(19t+2i-1)/2} u_i^{(23t+2i+1)/2}$$

Based the construction,  $B_{21t+2i+1}$  contains exactly three cycles of lengths:

$$21t + 2i + 1, 23t + 2i, 25t + 2i.$$

For  $0 \leq i \leq \frac{t-3}{2}$ , let the subgraph  $B_{21t+2i}$  consists of a cycle

$$xv_i^1 v_i^2 \dots v_i^{25t+2i} x$$

and a path:

$$xv_{i,1}^1 v_{i,1}^2 \dots v_{i,1}^{9t+i-1} v_i^{12t+i}$$

Based the construction,  $B_{21t+2i}$  contains exactly three cycles of lengths:

$$21t + 2i, 22t + 2i + 1, 25t + 2i + 1.$$

For  $i = \frac{t-1}{2}$ ,  $B_{21t+2i}$  is simply a cycle of length  $22t - 1$ .

For  $0 \leq i \leq \frac{t-3}{2}$ , let the subgraph  $B_{23t+2i+1}$  consists of a cycle

$$xw_i^1w_i^2\dots w_i^{26t+2i+1}x$$

and a path:

$$xw_{i,1}^1w_{i,1}^2\dots w_{i,1}^{(21t+2i-1)/2}w_i^{(25t+2i+1)/2}$$

Based the construction,  $B_{23t+2i+1}$  contains exactly three cycles of lengths:

$$23t + 2i + 1, 24t + 2i + 2, 26t + 2i + 2.$$

For  $i = \frac{t-1}{2}$ ,  $B_{23t+2i+1}$  is simply a cycle of length  $24t$ .

For  $58 \leq i \leq t - 742$ , let the subgraph  $B_{27t+i-57}$  consists of a cycle

$$C_{27t+i-57} = xy_i^1y_i^2\dots y_i^{132t+11i+893}x$$

and ten paths sharing a common vertex  $x$ , the other end vertices are on the cycle  $C_{27t+i-57}$ :

$$\begin{aligned} & xy_{i,1}^1y_{i,1}^2\dots y_{i,1}^{(17t-1)/2}y_i^{(37t-115)/2+i} \\ & xy_{i,2}^1y_{i,2}^2\dots y_{i,2}^{(19t-1)/2}y_i^{(57t-103)/2+2i} \\ & xy_{i,3}^1y_{i,3}^2\dots y_{i,3}^{(19t-1)/2}y_i^{(77t+315)/2+3i} \\ & xy_{i,4}^1y_{i,4}^2\dots y_{i,4}^{(21t-1)/2}y_i^{(97t+313)/2+4i} \\ & xy_{i,5}^1y_{i,5}^2\dots y_{i,5}^{(21t-1)/2}y_i^{(117t+313)/2+5i} \\ & xy_{i,6}^1y_{i,6}^2\dots y_{i,6}^{(23t-1)/2}y_i^{(137t+311)/2+6i} \\ & xy_{i,7}^1y_{i,7}^2\dots y_{i,7}^{(23t-1)/2}y_i^{(157t+309)/2+7i} \\ & xy_{i,8}^1y_{i,8}^2\dots y_{i,8}^{(25t-1)/2}y_i^{(177t+297)/2+8i} \\ & xy_{i,9}^1y_{i,9}^2\dots y_{i,9}^{(25t-1)/2}y_i^{(197t+301)/2+9i} \\ & xy_{i,10}^1y_{i,10}^2\dots y_{i,10}^{(27t-1)/2}y_i^{(217t+305)/2+10i}. \end{aligned}$$

As a cycle with  $d$  chords contains  $\binom{d+2}{2}$  distinct cycles,  $B_{27t+i-57}$  contains exactly 66 cycles of lengths:

$$\begin{aligned}
&27t + i - 57, & 28t + i + 7, & 29t + i + 210, & 30t + i, \\
&31t + i + 1, & 32t + i, & 33t + i, & 34t + i - 5, \\
&35t + i + 3, & 36t + i + 3, & 37t + i + 742, & 38t + 2i - 51, \\
&38t + 2i + 216, & 40t + 2i + 209, & 40t + 2i, & 42t + 2i, \\
&42t + 2i - 1, & 44t + 2i - 6, & 44t + 2i - 3, & 46t + 2i + 5, \\
&46t + 2i + 744, & 48t + 3i + 158, & 49t + 3i + 215, & 50t + 3i + 209, \\
&51t + 3i - 1, & 52t + 3i - 1, & 53t + 3i - 7, & 54t + 3i - 4, \\
&55t + 3i - 1, & 56t + 3i + 746, & 59t + 4i + 157, & 59t + 4i + 215, \\
&61t + 4i + 208, & 61t + 4i - 2, & 63t + 4i - 7, & 63t + 4i - 5, \\
&65t + 4i - 2, & 65t + 4i + 740, & 69t + 5i + 157, & 70t + 5i + 214, \\
&71t + 5i + 207, & 72t + 5i - 8, & 73t + 5i - 5, & 74t + 5i - 3, \\
&75t + 5i + 739, & 80t + 6i + 156, & 80t + 6i + 213, & 82t + 6i + 201, \\
&82t + 6i - 6, & 84t + 6i - 3, & 84t + 6i + 738, & 90t + 7i + 155, \\
&91t + 7i + 207, & 92t + 7i + 203, & 93t + 7i - 4, & 94t + 7i + 738, \\
&101t + 8i + 149, & 101t + 8i + 209, & 103t + 8i + 205, & 103t + 8i + 737, \\
&111t + 9i + 151, & 112t + 9i + 211, & 113t + 9i + 946, & 122t + 10i + 153, \\
&122t + 10i + 952, & 132t + 11i + 894.
\end{aligned}$$

$B_0$  is a path with an end vertex  $x$  and length  $n - n_t$ . Other  $B_i$  is simply a cycle of length  $i$ .

Then  $f(n) \geq n + \frac{107}{3}t + \frac{7}{3}$ , for  $n \geq n_t$ . This completes the proof.

From Theorem 7, we have

$$\liminf_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} \geq \sqrt{2 + \frac{7654}{19071}},$$

which is better than the previous bounds  $\sqrt{2}$  (see [15]),  $\sqrt{2 + \frac{2}{5}}$  (see [9]).

So Conjecture 4 is not true.

Combining this with Boros, Caro, Füredi and Yuster's upper bound, namely

Theorem 2, we get

$$1.98 \geq \limsup_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} \geq \liminf_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} \geq \sqrt{2 + \frac{7654}{19071}}.$$

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